QED and QCD
Tutorial Questions
Sheet 1

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Note: focus on Q1, Q2, Q3 (ii), one of the identities in Q4, and Q5.

Tutorial Questions 1: Klein-Gordon equation

Consider a plane-wave solution to the Klein-Gordon equation

$$\phi(\vec{x}, t) = A\,\mathrm{e}^{-i(\omega t - \vec{k}\cdot\vec{x})}\,. \tag{1}$$

a) Use the Klein-Gordon equation to find the dispersion relation, i.e. find ω as a function of $\vec{k}$. How do you interpret the two solutions?

b) Show that these solutions are eigenstates of the energy operator, $i\partial_t$, and the 3-momentum operator, $-i\nabla$.

Tutorial Questions 2: γ matrix identities

a) Show that the Hermitian conjugate of the γ matrices are

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \tag{2}$$

b) Show the following using the anti-commutation relations for the γ matrices in 4 dimensions

$$\gamma^\mu\gamma_\mu = 4\,, \tag{3}$$

$$\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu\,, \tag{4}$$

$$\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\mu = 4g^{\nu\lambda}\,, \tag{5}$$

$$\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho\gamma_\mu = -2\gamma^\rho\gamma^\lambda\gamma^\nu\,. \tag{6}$$

How does this change if $g^{\mu\nu}g_{\mu\nu} = d$ with $d \neq 4$?

Tutorial Questions 3: Orthonormality and completeness

Verify the orthonormality

$$u_r(p)^\dagger\gamma^0 u_s(p) = 2m\delta_{rs}\,, \qquad v_r(p)^\dagger\gamma^0 v_s(p) = -2m\delta_{rs}\,, \qquad u_r(p)^\dagger\gamma^0 v_s(p) = v_s(p)^\dagger\gamma^0 u_r(p) = 0\,, \tag{7}$$

and completeness relations

$$\sum_{r=1,2} u_r(p)u_r(p)^\dagger\gamma^0 = \gamma\cdot p + m \qquad \text{and} \qquad \sum_{r=1,2} v_r(p)v_r(p)^\dagger\gamma^0 = \gamma\cdot p - m\,. \tag{8}$$

– please turn over –

Tutorial Questions 4: Angular momentum and spin

a) Show that the Dirac Hamiltonian $H = \vec{\alpha} \cdot \vec{p} + \beta m$ commutes with the total angular momentum operator $\vec{L} + \vec{S}$ where $\vec{L} = \vec{x} \times \vec{p}$ and $\vec{S}$ is

$$\vec{S} = \frac{1}{2} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} . \tag{9}$$

b) Use the plane-wave solution to show that for $\vec{p} = (0, 0, p_z)$

$$S_z u_1 = \frac{1}{2} u_1 \,, \qquad S_z u_2 = -\frac{1}{2} u_2 \,, \qquad S_z v_1 = \frac{1}{2} v_1 \,, \quad \text{and} \quad S_z v_2 = -\frac{1}{2} v_2 \,, \tag{10}$$

where S_z is the z-component of the spin operator.