QED and QCD
Tutorial Questions
Sheet 1

2026
Dr Yannick Ulrich

`https://stfc.yannickulrich.com` Issued: 8 September 2026

Note: focus on Q1, Q2a, Q2b.ii, one of the identities in Q3b, and Q4a.

Tutorial Questions 1: Klein-Gordon equation

Consider a plane-wave solution to the Klein-Gordon equation

$$\phi(\vec{x},t) = A\,\mathrm{e}^{-i(\omega t - \vec{k}\cdot\vec{x})}\,. \tag{1}$$

a) Use the Klein-Gordon equation to find the dispersion relation, i.e. find ω as a function of $\vec{k}$. How do you interpret the two solutions?

b) Show that these solutions are eigenstates of the energy operator, $i\partial_t$, and the 3-momentum operator, $-i\nabla$.

SOLUTION:

a) Since $\phi(\vec{x},t)$ is a solution of the Klein-Gordon equation we must have

$$(\partial_t^2 - \nabla^2 + m^2)\phi = (-\omega^2 + \vec{k}^2 + m^2)\phi = 0\,,$$

which is satisfied if and only if

$$\omega(\vec{k}) = \pm\sqrt{\vec{k}^2 + m^2}\,.$$

For a fixed value of $\vec{k}$, the two solutions represent plane waves moving in opposite directions.

b) For the energy operator we have

$$i\partial_t\phi = \omega\phi = \pm\omega(\vec{k})\phi\,, \tag{2}$$

while for the momentum operator we have

$$-i\nabla\phi = \vec{k},\phi\,. \tag{3}$$

Tutorial Questions 2: γ matrix identities

a) Show that the Hermitian conjugate of the γ matrices are

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \tag{4}$$

b) Show the following using the anti-commutation relations for the γ matrices in 4 dimensions

$$\gamma^\mu\gamma_\mu = 4\,, \tag{5}$$
$$\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu\,, \tag{6}$$
$$\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\mu = 4g^{\nu\lambda}\,, \tag{7}$$
$$\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho\gamma_\mu = -2\gamma^\rho\gamma^\lambda\gamma^\nu\,. \tag{8}$$

How does this change if $g^{\mu\nu}g_{\mu\nu} = d$ with $d \neq 4$?

SOLUTION:

a) The matrix

$$\gamma^0 = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,,$$

is hermitian and satisfies $(\gamma^0)^2 = 1$, hence

$$(\gamma^0)^\dagger = \gamma^0 = \gamma^0(\gamma^0)^2 = \gamma^0\gamma^0\gamma^0\,.$$

The matrix

$$\gamma^i = \gamma^0\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\,.$$

is anti-hermitian, that is $(\gamma^i)^\dagger = -\gamma^i$. Using the commutation relation $\{\gamma^0, \gamma^i\} = 0$ we find

$$(\gamma^i)^\dagger = -\gamma^i = -\gamma^i(\gamma^0)^2 = \gamma^0\gamma^i\gamma^0\,.$$

b) In $d = 4$ we have $g^\mu_\mu = 4$

$$\begin{aligned}
\gamma^\mu\gamma_\mu &= \frac{1}{2}\{\gamma^\mu, \gamma_\mu\} = g^\mu_\mu = 4\,,\\
\gamma^\mu\gamma^\nu\gamma_\mu &= \{\gamma^\mu, \gamma^\nu\}\gamma_\mu - \gamma^\nu\gamma^\mu\gamma_\mu = 2g^{\mu\nu}\gamma_\mu - 4\gamma^\nu = -2\gamma^\nu\,,\\
\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\mu &= \{\gamma^\mu, \gamma^\nu\}\gamma^\lambda\gamma_\mu - \gamma^\nu\gamma^\mu\gamma^\lambda\gamma_\mu = 2\gamma^\lambda\gamma^\nu + 2\gamma^\nu\gamma^\lambda = 2\{\gamma^\nu, \gamma^\lambda\} = 4g^{\nu\lambda}\,,\\
\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho\gamma_\mu &= \{\gamma^\mu\gamma^\nu\}\gamma^\lambda\gamma^\rho\gamma_\mu - \gamma^\nu\gamma^\mu\gamma^\lambda\gamma^\rho\gamma_\mu = 2\gamma^\lambda\gamma^\rho\gamma^\nu - 4g^{\lambda\rho}\gamma^\nu = -2\gamma^\rho\gamma^\lambda\gamma^\nu\,.
\end{aligned}$$

In $d \neq 4$ we have $g^\mu_\mu = d$

$$\begin{aligned}
\gamma^\mu\gamma_\mu &= d\,,\\
\gamma^\mu\gamma^\nu\gamma_\mu &= (2-d)\gamma^\nu\,,\\
\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\mu &= 4g^{\nu\lambda} + (d-4)\gamma^\nu\gamma^\lambda\,,\\
\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho\gamma_\mu &= -2\gamma^\rho\gamma^\lambda\gamma^\nu - (d-4)\gamma^\nu\gamma^\lambda\gamma^\rho\,.
\end{aligned}$$

Tutorial Questions 3: Orthonormality and completeness

Verify

a) the orthonormality

$$u_r(p)^\dagger\gamma^0 u_s(p) = 2m\delta_{rs}\,, \qquad v_r(p)^\dagger\gamma^0 v_s(p) = -2m\delta_{rs}\,, \qquad u_r(p)^\dagger\gamma^0 v_s(p) = v_s(p)^\dagger\gamma^0 u_r(p) = 0\,, \tag{9}$$

b) and completeness relations

$$\sum_{r=1,2} u_r(p)u_r(p)^\dagger\gamma^0 = \gamma\cdot p + m \qquad \text{and} \qquad \sum_{r=1,2} v_r(p)v_r(p)^\dagger\gamma^0 = \gamma\cdot p - m\,. \tag{10}$$

SOLUTION:

a) We start from our definition of $u_s(\vec{p})$:

$$u_s(\vec{p}) = \sqrt{E+m}\begin{pmatrix}\chi_s\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s\end{pmatrix} \quad\Rightarrow\quad u_s^\dagger(\vec{p}) = \sqrt{E+m}\begin{pmatrix}\chi_s^\dagger & \chi_s^\dagger\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\end{pmatrix}.$$

This gives

$$\begin{aligned}\bar{u}_r(\vec{p})u_s(\vec{p}) &= (E+m)\begin{pmatrix}\chi_r^\dagger & \chi_r^\dagger\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\chi_s\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s\end{pmatrix}\\ &= (E+m)\chi_r^\dagger\chi_s - \chi_r^\dagger\frac{(\vec{\sigma}\cdot\vec{p})^2}{E+m}\chi_s\,.\end{aligned}$$

Using $\{\sigma_i,\sigma_j\} = 2\delta_{ij}$, we find $(\vec{\sigma}\cdot\vec{p})^2 = \vec{p}^2 = (E^2-m^2)$. This gives

$$\bar{u}_r(\vec{p})u_s(\vec{p}) = \Big(E+m-(E-m)\Big)\chi_r^\dagger\chi_s = 2m\delta_{rs}\,.$$

Similarly

$$\begin{aligned}\bar{v}_r(\vec{p})v_s(\vec{p}) &= (E+m)\begin{pmatrix}\chi_r^\dagger\frac{\vec{\sigma}\cdot\vec{p}}{E+m} & \chi_r^\dagger\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s\\ \chi_s\end{pmatrix}\\ &= -2m\chi_r^\dagger\chi_s = -2m\delta_{rs}\,,\end{aligned}$$

and

$$\begin{aligned}\bar{u}_r(\vec{p})v_s(\vec{p}) &= (E+m)\begin{pmatrix}\chi_r^\dagger\,, & \chi_r^\dagger\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s\\ \chi_s\end{pmatrix} = 0\,,\\ \bar{v}_r(\vec{p})u_s(\vec{p}) &= (E+m)\begin{pmatrix}\chi_r^\dagger\frac{\vec{\sigma}\cdot\vec{p}}{E+m} & \chi_r^\dagger\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\chi_s\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s\end{pmatrix} = 0\,.\end{aligned}$$

b) For the completeness relations we have

$$\begin{aligned}u_r(\vec{p})\bar{u}_r(\vec{p}) &= (E+m)\underbrace{\begin{pmatrix}\chi_r\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_r\end{pmatrix}\begin{pmatrix}\chi_r^\dagger & \chi_r^\dagger\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\end{pmatrix}}_{4\times 4\text{ matrix}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\\ &= (E+m)\begin{pmatrix}\chi_r\chi_r^\dagger & -\chi_r\chi_r^\dagger\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_r\chi_r^\dagger & -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_r\chi_r^\dagger\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\end{pmatrix}.\end{aligned}$$

Since χ_r, $r = 1, 2$, constitute a complete set, we have $\sum_{r=1,2} \chi_r \chi_r^\dagger = 1_2$, hence

$$\sum_{r=1,2} u_r(\vec{p})\bar{u}_r(\vec{p}) = \begin{pmatrix} E+m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -\frac{\vec{p}^2}{E+m} \end{pmatrix} = \begin{pmatrix} E+m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E+m \end{pmatrix} = \gamma \cdot p + m\,.$$

Similarly

$$\sum_{r=1,2} v_r(\vec{p})\bar{u}_r(\vec{p}) = \begin{pmatrix} E-m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -(E+m) \end{pmatrix} = \gamma \cdot p - m\,.$$

Tutorial Questions 4: Angular momentum and spin

a) Show that the Dirac Hamiltonian $H = \vec{\alpha}\cdot\vec{p} + \beta m$ commutes with the total angular momentum operator $\vec{L} + \vec{S}$ where $\vec{L} = \vec{x}\times\vec{p}$ and $\vec{S}$ is

$$\vec{S} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} . \tag{11}$$

b) Use the plane-wave solution to show that for $\vec{p} = (0, 0, p_z)$

$$S_z u_1 = \frac{1}{2}u_1 , \qquad S_z u_2 = -\frac{1}{2}u_2 , \qquad S_z v_1 = \frac{1}{2}v_1 , \quad \text{and} \quad S_z v_2 = -\frac{1}{2}v_2 , \tag{12}$$

where S_z is the z-component of the spin operator.

SOLUTION:

a) First we observe that $[\vec{L}, \beta m] = [\vec{S}, \beta m] = 0$. Second, since $\vec{p}$ is a vector under rotations, we have $[L_i, p_j] = i\varepsilon_{ijk}p_k$, giving $[\vec{L}, H] = i\vec{\alpha}\times\vec{p}$. The same can be verified with a direct computation using $L_i = i\varepsilon_{ijk}x_j p_k$, as well as the canonical commutation relations $[x_i, p_j] = i\delta_{ij}$, and $[x_i, x_j] = [p_i, p_j] = 0$. Last, using

$$[S_i, \alpha_j] = \frac{1}{2}\begin{pmatrix} 0 & [\sigma_i, \sigma_j] \\ [\sigma_i, \sigma_j] & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\varepsilon_{ijk}\sigma_k \\ i\varepsilon_{ijk}\sigma_k & 0 \end{pmatrix} = i\varepsilon_{ijk}\alpha_k ,$$

we find

$$[\vec{S}, H] = i\,\vec{p}\times\vec{\alpha} = -i\,\vec{\alpha}\times\vec{p} \quad \Rightarrow \quad [\vec{L} + \vec{S}, H] = 0 .$$

b) Since $\vec{p} = (0, 0, p_z)$ we have $\vec{\sigma}\cdot\vec{p} = \sigma_3 p_z$. This gives

$$u_1(\vec{p}) = \sqrt{E+m}\begin{pmatrix} \chi_1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_1 \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \frac{1}{2}\frac{p_z}{E+m}\chi_1 \end{pmatrix} ,$$

$$u_2(\vec{p}) = \begin{pmatrix} \chi_2 \\ -\frac{1}{2}\frac{p_z}{E+m}\chi_2 \end{pmatrix} .$$

This gives immediately

$$S_z u_1 = \frac{1}{2}u_1 , \qquad S_z u_2 = -\frac{1}{2}u_2 .$$

Similarly

$$v_1(\vec{p}) = \sqrt{E+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_1 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\frac{p_z}{E+m}\chi_1 \\ \chi_1 \end{pmatrix} ,$$

$$u_2(\vec{p}) = \begin{pmatrix} -\frac{1}{2}\frac{p_z}{E+m}\chi_2 \\ \chi_2 \end{pmatrix} .$$

This gives immediately

$$S_z v_1 = \frac{1}{2}v_1 , \qquad S_z v_2 = -\frac{1}{2}v_2 .$$