QED and QCD
Tutorial Questions
Sheet 2

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Note: if pressed for time ignore the "arbitrary gauge" part in the first question. The second question is partly a repetition of what was already presented in the lectures. You can focus on a and b

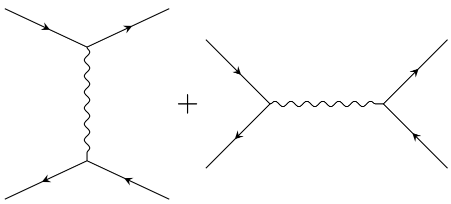
Tutorial Questions 1: Bhabha scattering

Draw all the tree-level diagrams for Bhabha-scattering

$$e^+(p_1)e^-(p_2) \to e^+(p_3)e^-(p_4)\,, \tag{1}$$

and give the expression for the scattering amplitude in Feynman gauge. What happens in an arbitrary gauge?

SOLUTION: We have two diagrams

$\mathcal{M} =$ 

$$= (ie)^2\Big[\bar v(p_1)\gamma^\mu v(p_3)\Big] \times \Big[\bar u(p_4)\gamma_\mu u(p_2)\Big] \times \frac{-i}{(p_1-p_3)^2} \times (-1)$$
$$+ (ie)^2\Big[\bar v(p_1)\gamma^\mu u(p_2)\Big] \times \Big[\bar u(p_4)\gamma_\mu v(p_3)\Big] \times \frac{-i}{(p_1+p_2)^2}\,.$$

The relative minus sign between the diagrams arises from the anti-fermion line.

In general gauge, the propagator becomes

$$\frac{-i}{q^2}g^{\mu\nu} \to \frac{i}{q^2}\Big(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\Big)$$

However, the terms proportional to $q^\mu q^\nu$ vanish, so that the amplitude is the same as in the Feynman gauge. For instance, let us consider a term proportional to $(p_1-p_3)^\mu$ in the first line. Due to Dirac equation such a term vanishes

$$\big[\bar v(p_1)\gamma^\mu v(p_3)\big](p-p')_\mu = \bar v(p_1)(\gamma\cdot p_1 - \gamma\cdot p_3)v(p_3)$$
$$= \bar v(p_1)\big[(\gamma\cdot p_1 - m) - (\gamma\cdot p_3 - m)\big]v(p_3) = 0\,.$$

Tutorial Questions 2: Muon production

Let us consider

$$e^+(p_1)e^-(p_2) \to \mu^+(p_3)\mu^-(p_4)\,, \tag{2}$$

in the limit $m_e = 0$.

a) Show that the matrix element squared is

$$\sum |\mathcal{M}|^2 = \frac{e^4}{(p_1+p_2)^4} \mathrm{tr}\Big[\gamma\cdot p_1\,\gamma^\mu\,\gamma\cdot p_2\,\gamma^\nu\Big]\mathrm{tr}\Big[(\gamma\cdot p_4 + M)\,\gamma_\mu\,(\gamma\cdot p_3 - M)\,\gamma_\nu\Big]\,, \tag{3}$$

when summed over spins.

b) Show that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}\sqrt{1-\frac{4M^2}{s}}\,\frac{1}{4}\sum|\mathcal{M}|^2\,, \tag{4}$$

with $s = (p_1+p_2)^2$.

c) If you have time, evaluate the traces to show

$$\sum |\mathcal{M}|^2 = \frac{2e^4}{s^2}\Big[(p_4\cdot p_2)^2 + (p_4\cdot p_1)^2 + M^2(p_1\cdot p_2)\Big]\,. \tag{5}$$

d) Show that, in the centre-of-mass frame with scattering angle θ,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{e^4}{64\pi^2 s}\sqrt{1-\frac{4M^2}{s}}\left[1+\left(1-\frac{4M^2}{s}\right)\cos^2\theta + \frac{4M^2}{s}\right] \tag{6}$$

e) Find an expression for the total cross section in the high-energy limit where the mass of the muon can be neglected.

SOLUTION:

a) We can adapt the result of the previous exercise or from the lecture

$$\mathcal{M} = +(ie)^2\Big[\bar{v}(p_1)\gamma^\mu u(p_2)\Big] \times \Big[\bar{u}(p_4)\gamma_\mu v(p_3)\Big] \times \frac{-i}{(p_1+p_2)^2}\,.$$

Squaring the amplitude results in

$$\begin{aligned}
\sum |\mathcal{M}|^2 &= \frac{e^4}{(p_1+p_2)^2}\Big[\bar{v}(p_1)\gamma^\mu u(p_2)\Big]\Big[\bar{u}(p_4)\gamma_\mu v(p_3)\Big]\Big[\bar{u}(p_2)\gamma^\nu v(p_1)\Big]\Big[\bar{v}(p_3)\gamma_\nu u(p_4)\Big] \\
&= \frac{e^4}{(p_1+p_2)^2}\Big[\bar{v}(p_1)\gamma^\mu u(p_2)\bar{u}(p_2)\gamma^\nu v(p_1)\Big]\Big[\bar{u}(p_4)\gamma_\mu v(p_3)\bar{v}(p_3)\gamma_\nu u(p_4)\Big] \\
&= \frac{e^4}{(p_1+p_2)^2}\mathrm{tr}\Big[\gamma\cdot p_1\,\gamma^\mu\,\gamma\cdot p_2\,\gamma^\nu\Big]\mathrm{tr}\Big[(\gamma\cdot p_4 + M)\,\gamma_\mu\,(\gamma\cdot p_3 - M)\,\gamma_\nu\Big]\,.
\end{aligned}$$

b) In the centre-of-mass frame, due to momentum conservation we have $\vec{p}_1 = -\vec{p}_2$ and $\vec{p}_3 = -\vec{p}_4$. From this it follows that $E_1 = E_2 = E_3 = E_4 = \sqrt{s}/2$. Using the result for $\mathrm{d}\Phi$ from the lecture notes, we have

$$\mathrm{d}\Phi = \frac{1}{16\pi^2}\frac{|\vec{p}_3|}{\sqrt{s}}\mathrm{d}\Omega\,,$$

with $|\vec{p}_3| = \sqrt{E_3^2 - M^2} = \sqrt{s/4 - M^2}$. The flux $\mathcal{F}$ evaluates to with $m_e = 0$

$$\mathcal{F} = \frac{1}{4p_1\cdot p_2} = \frac{1}{2s}\,.$$

Therefore,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}\sqrt{1-\frac{4M^2}{s}}\,.$$

c) We for the muon trace

$$\begin{aligned}\mathrm{tr}\Big[\cdots\Big] &= p_3^\rho p_4^\sigma \mathrm{tr}\Big[\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu\Big] - M\ \mathrm{tr}\Big[\gamma\cdot p_3\ \gamma_\mu\ \gamma_\nu\Big] + M\ \mathrm{tr}\Big[\gamma_\mu\ \gamma\cdot p_4\ \gamma_\nu\Big] - M^4\mathrm{tr}\Big[\gamma_\mu\gamma_\nu\Big]\\ &= p_3^\rho p_4^\sigma 4\Big(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\sigma}g_{\mu\nu} + g_{\rho\nu}g_{\mu\sigma}\Big) - M^4 4 g_{\mu\nu}\\ &= 4p_{3,\mu}p_{4,\nu} + 4p_{3,\nu}p_{4,\mu} - 4(p_3\cdot p_4 + M^4)g_{\eta\mu\nu}\,.\end{aligned}$$

For the electron trace we find similarly

$$\mathrm{tr}\Big[\cdots\Big] = 4p_1^\mu p_2^\nu + 4p_1^\nu p_2^\mu - 4(p_1\cdot p_2)g^{\mu\nu}\,.$$

Multiplying this out and using momentum conservation results in

$$\sum|\mathcal{M}|^2 = \frac{2e^4}{s^2}\Big[(p_4\cdot p_2)^2 + (p_4\cdot p_1)^2 + M^2(p_1\cdot p_2)\Big]\,. \tag{7}$$

Here we have used that

$$\begin{aligned}p_1^2 = p_2^2 = 0\,,\qquad p_3^2 = p_4^2 = M^2\,,\qquad p_3\cdot p_4 = p_1\cdot p_2 - M^2\,,\\ p_1\cdot p_3 = p_2\cdot p_4\,,\qquad p_2\cdot p_3 = p_1\cdot p_4\,.\end{aligned} \tag{8}$$

d) We first need to identify scalar products

$$p_1\cdot p_2 = s/2\,,\qquad p_2\cdot p_4 = \frac{s}{4}(1-\beta\cos\theta)\,,\qquad p_1\cdot p_4 = \frac{s}{4}(1+\beta\cos\theta)\,.$$

Here we have defined $\beta = \sqrt{1-4M^2/s}$. Therefore,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{e^4}{64\pi^2 s}\beta\Big[1+\beta^2\cos^2\theta + \frac{4M^2}{s}\Big]\,.$$

e) In the high-energy limit we have $\beta = 1$ and $M = 0$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{e^4}{64\pi^2 s}\Big[1+\cos^2\theta\Big]\,.$$

With $\mathrm{d}\Omega = \mathrm{d}\phi\mathrm{d}(\cos\theta)$,

$$\sigma = \int \mathrm{d}\phi\mathrm{d}(\cos\theta)\frac{e^4}{64\pi^2 s}\Big[1+\cos^2\theta\Big] = \int_{-1}^{1}\mathrm{d}(\cos\theta)\frac{e^4}{32\pi s}\Big[1+\cos^2\theta\Big] = \frac{e^4}{12\pi s} = \frac{4\pi\alpha^2}{3s}\,.$$