QED and QCD
Tutorial Questions
Sheet 3

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Dr Yannick Ulrich

`https://stfc.yannickulrich.com`

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Note: focus on Q1a, Q1b and Q2.

Tutorial Questions 1: Compton scattering Consider Compton scattering

$$e(p_1)\gamma(p_2) \to e(p_3)\gamma(p_4)\,. \tag{1}$$

a) Derive the amplitude for Compton scattering and verify that it is gauge-invariant.

b) Using the matrix element squared from the lecture

$$\sum|\mathcal{M}|^2 = 8e^4\left(\frac{p_1\cdot p_2}{p_1\cdot p_4}+\frac{p_1\cdot p_4}{p_1\cdot p_2}+2m^2\left(\frac{1}{p_1\cdot p_2}-\frac{1}{p_1\cdot p_4}\right)+m^4\left(\frac{1}{p_1\cdot p_2}-\frac{1}{p_1\cdot p_4}\right)^2\right). \tag{2}$$

Working in the centre-of-mass system, in the limit where the electron mass m can be neglected, show that the matrix element squared is dominated by backward scattering, $\theta\simeq\pi$, where θ is the scattering angle of the photon.

c) Use the matrix-element squared for Compton scattering to obtain the matrix-element squared for the annihilation process $ee\to\gamma\gamma$. Again work in the centre-of-mass frame and show that, in the high-energy limit $E\gg m$

$$\sum|\mathcal{M}|^2 = 16e^4\frac{1+\cos^2\theta}{\sin^2\theta}\,. \tag{3}$$

SOLUTION:

a) The amplitude was written down in the lecture notes.

$$\mathcal{M} = -ie^2\epsilon^{*\mu}(p_4)\,\epsilon^\nu(p_2)\ \bar u(p_3)\left(\gamma_\mu\frac{\gamma\cdot(p_1+p_2)+m}{(p_1+p_2)^2-m^2}\gamma_\nu+\gamma_\nu\frac{\gamma\cdot(p_1-p_4)+m}{(p_1-p_4)^2-m^2}\gamma_\mu\right)u(p_1)\,.$$

To verify gauge invariance, let us contract with p_4^μ or p_2^ν

$$\begin{aligned}
p_2^\nu\Big[\cdots\Big] &= \bar u(p_3)\left(\gamma_\mu\frac{\gamma\cdot(p_1+p_2)+m}{(p_1+p_2)^2-m^2}\gamma\cdot p_2+\gamma\cdot p_2\frac{\gamma\cdot(p_1-p_4)+m}{(p_1-p_4)^2-m^2}\gamma_\mu\right)u(p_1)\\
&= \bar u(p_3)\left(\gamma_\mu\frac{\gamma\cdot p_1+\gamma\cdot p_2)+m}{(p_1+p_2)^2-m^2}\gamma\cdot p_2+\gamma\cdot p_2\frac{\gamma\cdot(p_2-p_3)+m}{(p_2-p_3)^2-m^2}\gamma_\mu\right)u(p_1)\\
&= \bar u(p_3)\left(\gamma_\mu\frac{\gamma\cdot p_1+m}{(p_1+p_2)^2-m^2}\gamma\cdot p_2+\gamma\cdot p_2\frac{-\gamma\cdot p_3+m}{(p_2-p_3)^2-m^2}\gamma_\mu\right)u(p_1)\,,
\end{aligned}$$

where we have used that $(\gamma \cdot p_2)(\gamma \cdot p_2) = p_2^2 = 0$ for the first term and have re-written $p_1 - p_4 = p_2 - p_3$ for the second. We can now use the anticommutation relation $\{\gamma \cdot p_1, \gamma \cdot p_2\} = \gamma \cdot p_1 \, \gamma \cdot p_2 + \gamma \cdot p_2 \, \gamma \cdot p_1 = 2 p_1 \cdot p_2$ to write

$$p_2^\nu \Big[\cdots \Big] = \bar{u}(p_3) \left(\gamma_\mu \frac{2p_1 \cdot p_2 - \gamma \cdot p_2 (\gamma \cdot p_1 - m)}{(p_1 + p_2)^2 - m^2} - \frac{2p_3 \cdot p_2 - (\gamma \cdot p_3 - m) \gamma \cdot p_2}{(p_2 - p_3)^2 - m^2} \gamma_\mu \right) u(p_1) \, .$$

Next, we use the Dirac equation $\bar{u}(p_3)(\gamma \cdot p_3 - m) = (\gamma \cdot p_1 - m) u(p_1) = 0$

$$p_2^\nu \Big[\cdots \Big] = \bar{u}(p_3) \gamma_\mu u(p_1) \left(\frac{2p_1 \cdot p_2}{(p_1 + p_2)^2 - m^2} - \frac{2p_3 \cdot p_2}{(p_2 - p_3)^2 - m^2} \right) .$$

Finally, $(p_1 + p_2)^2 - m^2 = 2p_1 \cdot p_2$ and $(p_2 - p_3)^2 = -2p_2 \cdot p_3$

$$p_2^\nu \Big[\cdots \Big] = \bar{u}(p_3) \gamma_\mu u(p_1) \Big(1 - 1 \Big) = 0 \, .$$

We can use the exact same logic for the p_4^μ contraction.

b) We have $m = 0$ and $p_1 \cdot p_2 = s/2$ and $p_1 \cdot p_4 = s/4(1 + \cos\theta)$. Therefore,

$$\sum |\mathcal{M}|^2 = 8e^4 \left(\frac{p_1 \cdot p_2}{p_1 \cdot p_4} + \frac{p_1 \cdot p_4}{p_1 \cdot p_2} \right) = 8e^4 \left(\frac{2}{1 + \cos\theta} + \frac{1 + \cos\theta}{2} \right) .$$

This is dominated for the region where $\cos\theta \to -1$, i.e. $\theta \to \pi$.

c) We begin with

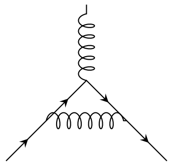
$$\sum |\mathcal{M}|^2 = 8e^4 \left(\frac{p_1 \cdot p_3}{p_1 \cdot p_4} + \frac{p_1 \cdot p_4}{p_1 \cdot p_3} \right) ,$$

and use additionally $p_1 \cdot p_3 = p_2 \cdot p_4 = s/4(1 - \cos\theta)$

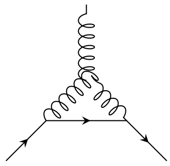
$$\sum |\mathcal{M}|^2 = 8e^4 \left(\frac{1 - \cos\theta}{1 + \cos\theta} + \frac{1 + \cos\theta}{1 - \cos\theta} \right) = 16e^4 \frac{1 + \cos^2\theta}{\sin^2\theta} \, .$$

Tutorial Questions 2: Colour factors

Show that the colour factors in the following diagrams are



$$= t^c t^a t^b \delta^{bc} = -\frac{1}{2N_c} t^a \,, \tag{4}$$



$$= i f^{abc} t^b t^c = -\frac{1}{2} C_A t^a \,. \tag{5}$$

SOLUTION:

a) Following the quark line backwards, we have

$$t^b t^a t^c \times \delta^{bc} = t^b t^a t^b = t^b \big(t^b t^a + [t^a, t^b] \big) = C_F t^a + i f^{abc} t^b t^c \,.$$

Further,

$$f^{abc} t^b t^c = f^{abc} \Big(\tfrac{1}{2} [t^b t^c] + \tfrac{1}{2} \{ t^b, t^c \} \Big) = \frac{i}{2} f^{abc} f^{bcd} t^d = \frac{i}{2} C_A t^a \,.$$

Here we have used that the anti-commutator is symmetric while f^{abc} is anti-symmetric under exchange of b and c. Therefore, their contraction vanishes. Finally,

$$t^b t^a t^c \times \delta^{bc} = C_F t^a - \frac{1}{2} C_A t^a = -\frac{1}{2N_c} t^a \,.$$

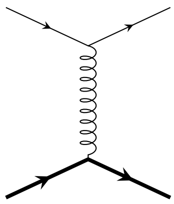
b)

$$t^b t^c i f^{abc} = -\frac{C_A}{2} t^c \,.$$

Tutorial Questions 3: Quark scattering

Calculate the spin-summed matrix element squared for quark scattering $ud \to ud$ in the limit of vanishing quark masses.

SOLUTION:

$\mathcal{M} =$  .

The Dirac algebra is identical to $e\mu \to e\mu$ while we get an additional colour factor

$$\mathcal{M} = \mathcal{M}_{e\mu\to e\mu} t^a_{ji} t^b_{lk} \delta^{ab} = \frac{ig_s^2}{(p_2 - p_4)^2 + i\epsilon} \Big[\bar{u}_{s_3}(p_3)\gamma^\mu u_{1_1}(p_1)\Big] \Big[\bar{u}_{s_4}(p_4)\gamma_\mu u_{s_2}(p_2)\Big] t^a_{ji} t^a_{lk} \,.$$

Squaring, we find

$$\sum |\mathcal{M}|^2 = \frac{8g_s^4}{t^2}\big(s^2 + u^2\big) \sum_{ijkl} t^a_{ji} t^a_{lk} t^b_{kl} t^b_{ij} \,.$$

We can calculate the colour trace

$$\sum_{ijkl} t^a_{ji} t^a_{lk} t^b_{kl} t^b_{ij} = \mathrm{tr}(t^a t^b)\mathrm{tr}(t^a t^b) = \frac{1}{4}\delta^{ab}\delta^{ab} = \frac{N_c^2 - 1}{4} \,.$$

Once all the prefactors from spin (1/2 for each incoming particle) and colour ($1/N_c$) for each incoming particle) are divided out, we have

$$\frac{1}{4N_c^2} \sum |\mathcal{M}|^2 = \frac{C_F}{2N_c} g_s^4 \frac{\big(s^2 + u^2\big)}{t^2} \,.$$